

On a classification of fat bundles over compact homogeneous spaces

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Abstract

This article deals with fat bundles. Bérard-Bergery classified all homogeneous bundles of that type. We ask a question of a possibility to generalize his description in the case of arbitrary G -structures over homogeneous spaces. We obtain necessary conditions for the existence of such bundles. These conditions yield a kind of classification of fat bundles associated with G -structures over compact homogeneous spaces provided that the connection in a G -structure is canonical.

1 Introduction

Let $P(M, G)$ be a principal bundle endowed with a connection form θ and its curvature form Ω . Let $\text{Ker } \theta = \mathcal{H} \subset TP$ be the corresponding horizontal distribution. Assume that the Lie algebra \mathfrak{g} of G is endowed with an invariant non-degenerate bilinear form $B_{\mathfrak{g}}$. We say that a vector $u \in \mathfrak{g}$ is *fat* or that the connection form θ is *u-fat*, if the bilinear two-form $B_{\mathfrak{g}}(\Omega(\cdot, \cdot), u)$ is non-degenerate on \mathcal{H} . If the fatness condition is fulfilled for every non-zero $u \in \mathfrak{g}$ then we say that the connection form θ is fat or that the principal bundle is fat. The role of the fatness condition in Riemannian geometry follows from its relation with the O'Neil tensor [2], [19] of a specific fiberwise metric on the associated bundle. In greater detail, consider an associated bundle

$$F \rightarrow P \times_G F \rightarrow M.$$

Endow F with a G -invariant Riemannian metric g_F , and M with a Riemannian metric g_M . Equip $P \times_G F$ with the *connection metric* defining it to be the Riemannian metric which equals g_F on F , $g(X^*, Y^*) = g_M(X, Y)$ for horizontal lifts X^*, Y^* of $X, Y \in TM$ respectively, and declaring TF and \mathcal{H} to be orthogonal with respect to g . For this metric the following holds. Let A_X denote the O'Neil tensor.

Theorem 1. [19] *The connection metric g on $P \times_G F$ is complete and defines a Riemannian submersion $\pi : P \times_G F \rightarrow M$ with totally geodesic fiber F and with holonomy group a subgroup of G . Conversely, every Riemannian submersion over M with totally geodesic fibers arises in this fashion. Moreover, the O'Neil tensor for such submersion satisfies the equality*

$$\theta(A_X Y) = -\Omega(X, Y).$$

The condition of fatness is an important tool of constructing manifolds of positive and non-negative curvature [9],[18]. In the Riemannian context, the following definition of fatness is used: a Riemannian submersion $\pi : E \rightarrow M$ with totally geodesic fibers is fat, if $A_X U \neq 0$ for all horizontal vector fields X and vertical vector fields U ("all vertical curvatures are positive"). For the associated bundles, the characterization of fatness of any connection is known [19] and can be formulated as follows.

Theorem 2. [19] *The Riemannian submersion $P \times_G F \rightarrow M$ with totally geodesic fibers G/L is fat if and only if the 2-form $B_{\mathfrak{g}}(\Omega(X, Y), u)$ is non-degenerate on the horizontal distribution for all $u \in \mathfrak{l}^\perp$.*

Keeping the above theorem in mind, from now on whenever we say that an associated bundle $G/L \rightarrow P/L \rightarrow M$ is fat, we mean that the set \mathfrak{l}^\perp consists of fat vectors. In this paper we consider associated bundles with homogeneous fibers since it was proved (see Proposition 2.6 in [19]) that every fat submersion necessarily has a homogeneous fiber. Note that the connection metric g depends on a (chosen) principal connection. The following is known.

1. There is an algebraic condition on the curvature tensor of the associated metric connection of the sphere bundles of the form

$$SO(n+1)/SO(n) \rightarrow P/SO(n) \rightarrow M$$

ensuring fatness (see Proposition 2.21 in [19]).

2. A theorem in [5] shows that the only fat $SO(4)/SO(3)$ -bundle over S^4 is the Hopf bundle $S^7 \rightarrow S^4$.
3. A theorem of Bérard-Bergery [3] which classifies all homogeneous fat bundles, that is, associated bundles of the form $H/L \rightarrow K/L \rightarrow K/H$, where K, H, L are compact Lie groups. Note that in this case the classification is obtained for *any* invariant connection.
4. There are necessary conditions for fatness, see for example [8].

To stress the fact that in general the fatness condition is dependent on the choice of the connection, we will always refer to “fatness with respect to a connection”. Here it seems to be instructive to compare [3] with the general problem. In the case of principal bundles $K \rightarrow K/H$, there is a one-to-one correspondence between the invariant connections and the linear maps $\Lambda : \mathfrak{k} \rightarrow \mathfrak{h}$ satisfying the conditions $\Lambda(X) = X$ for any $X \in \mathfrak{h}$, and $\Lambda([X, Y]) = [X, \Lambda(Y)]$ for all $X \in \mathfrak{h}, Y \in \mathfrak{k}$. If, for example, \mathfrak{k} is semisimple the fatness condition can be expressed as follows: for any $X \in \mathfrak{h}^\perp$ and any $Y \in \text{Ker } \Lambda$ there exists $Z \in \text{Ker } \Lambda$ such that $\langle X, \Lambda([Y, Z]) \rangle \neq 0$. Here $\langle \cdot, \cdot \rangle$ denotes the Killing form. The latter condition can be expressed entirely in terms of the Lie brackets of \mathfrak{k} . In general, the problem becomes much more complicated even for the invariant connections in G -structures. In this work we will adopt the following terminology. A homogeneous bundle

$$H/L \rightarrow K/L \rightarrow K/H$$

endowed with an invariant fat connection will be called the *Bérard-Bergery bundle*. The triples (K, H, L) (classified in [3]) will be called the *Bérard-Bergery triples*, and the pairs $(H, H \cap L)$ will be called the *Bérard-Bergery pairs*. Note that all (K, H, L) and $(H, H \cap L)$ are known [3] (see Theorem 2 in this paper). In this work we use this classification, therefore, we reproduce part of it in the last section. In this article we want to follow the line of reasoning of [3] and ask a more general question: *can one classify bundles associated with a G -structure over a compact homogeneous space K/H which are fat with respect to invariant connections?* The purpose of this paper is to solve this problem for the class of fat bundles over homogeneous spaces determined by the canonical invariant connections (Theorem 3). Our result yields necessary conditions, which can be considered as a kind of classification, because we show that the bases of such bundles are determined by the Bérard-Bergery triples, and they are determined

by a fiber bundle, which is naturally related to the Bérard-Bergery bundle. For example, if G is simple, there are no other fat bundles than the Bérard-Bergery homogeneous bundles. Consider the Bérard-Bergery pair $(H, H \cap L)$. Let \mathfrak{n}' be a maximal common ideal in \mathfrak{h} and $\mathfrak{h} \cap \mathfrak{l}$. Then, clearly,

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{n}', \mathfrak{h} \cap \mathfrak{l} = (\mathfrak{h} \cap \mathfrak{l})' \oplus \mathfrak{n}'.$$

Analogously, let \mathfrak{n} be a common maximal ideal in \mathfrak{g} and \mathfrak{l} .

Theorem 3. *Let $G \rightarrow P \rightarrow K/H$ be a G -structure over a homogeneous space K/H of a compact semisimple Lie group K . Assume that G is compact and semisimple. Let*

$$G/L \rightarrow P/L \rightarrow K/H$$

be an associated bundle. If it is fat with respect to the canonical connection, then:

1. $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ and $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{l})$ is a Bérard-Bergery pair;
2. K/H is a symmetric space of maximal rank;
3. $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{n}'$ and $\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{n}$

In particular, if G is simple, then the associated bundle is fat with respect to the canonical connection only if $G = H, P = K$. Therefore, the fiber bundle is a Bérard-Bergery bundle.

Let us mention a recent publication [7], where the authors consider a particular class of fat bundles over 4-manifolds M endowed with an S^1 -action which lifts to an $SO(3)$ -bundle and leaves the connection invariant. Thus, the case considered in [7] is an opposite to the case we deal with. It would be interesting to get deeper understanding of the symmetry of fat connections. Finally, let us mention that there is a different but related notion of symplectic fatness [18], and there is an interplay between these two [6],[4], [14].

2 Preliminaries

2.1 Notation, terminology and assumptions

The basic tools in this work are invariant connections and the Lie group theory. Therefore, we closely follow the terminology and notation in

[12],[13], [11] and [17] without further explanations. We denote the Lie algebras of Lie groups by the corresponding Gothic letters, so the Lie algebra of the Lie group G is denoted by \mathfrak{g} , etc. We consider principal bundles $G \rightarrow P \rightarrow M$ and in the most cases use the notation $P(M, G)$. Throughout the whole article we consider homogeneous spaces K/H and assume that both K and H are compact Lie groups. In this note we will use the following result.

Proposition 1. [3],[19]. *If $H/L \rightarrow K/L \rightarrow K/H$ is a fat bundle with $\dim(H/L) > 1$, then K/H is a Riemannian symmetric space and $\text{rank } K = \text{rank } H$.*

2.2 Invariant connections on reductive homogeneous spaces

In this subsection we will recall the theory of invariant connections on homogeneous spaces in the form presented in Sections 1 and 2 of Chapter X of [13]. Let M be a smooth manifold of dimension n , and let

$$G \rightarrow P \rightarrow M$$

be a G -structure, that is, a reduction of the frame bundle $L(M) \rightarrow M$ to a Lie group G . Any diffeomorphism $f \in \text{Diff}(M)$ acts on $L(M)$ by the formula

$$f(u) := (df_x X_1, \dots, df_x X_n)$$

for any frame $u = (X_1, \dots, X_n)$, $X_i \in T_x M$ over a point $x \in M$. By definition, f is called an automorphism of the given G -structure, if this action commutes with the action of G .

Let $M = K/H$ be a homogeneous space of a connected Lie group K . Assume that M is equipped with a K -invariant G -structure. The latter means that any left translation $\tau(k) : K/H \rightarrow K/H$, $\tau(k)(aH) = kaH$ lifts to an automorphism. Let $o = H \in K/H$. Consider the linear isotropy representation of H , that is, a homomorphism $H \rightarrow GL(T_o M)$ given by the formula

$$h \mapsto d\tau(h)_o, \text{ for } h \in H, o = H \in K/H.$$

It is important to observe that we can fix a frame $u_o : \mathbb{R}^n \rightarrow T_o M$, $u_o \in P$ and identify the linear isotropy representation of H with a homomorphism $\lambda : H \rightarrow G$

$$\lambda(h) = u_o^{-1} d\tau(h)_o u_o, h \in H.$$

One can see this as follows. Denote by $P_o \subset P$ the fiber over the point o , then $u_o \in P_o$. The action of H lifted to P preserves P_o , hence $h(u_o) \in P_o$. Since the structure group G acts transitively on P_o , there exists exactly one $g \in G$ such that

$$h(u_o) = u_o g.$$

It is easy to see that $\lambda(h) = g$.

In the sequel we assume that K/H is reductive. In this case \mathfrak{k} can be decomposed into a direct sum

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$$

such that $Ad_H(\mathfrak{m}) \subset \mathfrak{m}$. Note that the latter implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Note that since we consider compact Lie groups, the reductivity can be assumed (see, for example, [13], Chapter X). Also, we assume that the isotropy representation is faithful. Let us make one more straightforward but important observation. One can identify λ with the restriction of the adjoint representation of H on \mathfrak{m} (which we also denote by λ).

We say that a connection θ in $P \rightarrow M$ is K -invariant, if for any $k \in K$ the lift of $\tau(k)$ preserves it. We need the following description of the set of invariant connections in the principal bundle $P \rightarrow M$ from [13], Chapter X.

Theorem 4. *Let $M = K/H$ be a reductive homogeneous space equipped with a K -invariant G -structure. Then there is a one-to-one correspondence between the K -invariant connections in it, and Ad_H -invariant linear maps*

$$\Lambda : \mathfrak{k} \rightarrow \mathfrak{g}, \Lambda|_{\mathfrak{h}} = \lambda.$$

Let $\Lambda_{\mathfrak{m}} = \Lambda|_{\mathfrak{m}}$. Note that here Ad_H -invariance means that

$$\Lambda_{\mathfrak{m}}(Ad h(Z)) = \lambda(h)(\Lambda_{\mathfrak{m}}(Z)), Z \in \mathfrak{m}, h \in H.$$

Definition 1. Recall that a connection in the given K -invariant G -structure is called *canonical* if it corresponds to the map Λ with $\Lambda_{\mathfrak{m}} = 0$.

In this article we will need the description of the holonomy Lie algebra of the canonical connection.

Theorem 5. [13] *The holonomy Lie algebra of the canonical connection of a reductive homogeneous space K/H is given by the formula*

$$\mathfrak{h}^* = \lambda(ad([X, Y]_{\mathfrak{h}})), \forall X, Y \in \mathfrak{m}.$$

2.3 Factorizations of Lie groups and Lie algebras

Following Onishchik [15],[16] we say that a triple of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ is a *factorization*, if \mathfrak{h} and \mathfrak{l} are subalgebras of \mathfrak{g} , $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, and $\mathfrak{h} \neq \mathfrak{g}$, $\mathfrak{l} \neq \mathfrak{g}$. In the same way, we say that (G, H, L) is a factorization, if H and L are Lie subgroups of G , $G = H \cdot L$ and $G \neq H$, $G \neq L$. It is proved in [15],[16] that, for compact Lie groups, (G, H, L) is a factorization if and only if $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ is a factorization. All factorizations of simple compact Lie algebras are classified in [16]. The proof of our result is essentially based on this classification, and we reproduce it here. As usual, we consider the complexification \mathfrak{g}^c and a Cartan subalgebra $\mathfrak{j} \subset \mathfrak{g}^c$. Consider the corresponding root system Δ , and choose the set of simple roots $\alpha_1, \dots, \alpha_m$. It is known that any irreducible linear representation of \mathfrak{g} is determined (up to an equivalence) by the highest weight of the representation. This is a vector $\Phi \in \mathfrak{j}$, which can be described by the integers

$$\Phi_i = \frac{2\langle \Phi, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}, \quad i = 1, \dots, m.$$

We write $\Phi = (\Phi_1, \dots, \Phi_m)$ and denote by φ_i the irreducible representation of a simple Lie algebra \mathfrak{g} with the highest weight Φ which has $\Phi_i = 1$ and $\Phi_j = 0, i \neq j$. The following classification result is proved by Onishchik.

Theorem 6. [15] *Let \mathfrak{g} be a compact simple Lie algebra. All factorizations $(\mathfrak{g}, \mathfrak{l}, \mathfrak{h})$ and possible embeddings*

$$i' : \mathfrak{h} \rightarrow \mathfrak{g}, i'' : \mathfrak{l} \rightarrow \mathfrak{g}$$

are given in Table 1.

\mathfrak{g}	\mathfrak{h}	i'	\mathfrak{l}	i''	$\mathfrak{h} \cap \mathfrak{l}$	restrictions
A_{2n-1}	C_n	φ_1	A_{2n-2}	$\varphi_1 + N$	C_{n-1}	$n > 1$
A_{2n-1}	C_n	φ_1	$A_{2n-2} \oplus T$	$\varphi_1 + N$	$C_{n-1} \oplus T$	$n > 1$
B_3	G_2	φ_2	B_2	$\varphi_1 + 2N$	A_1	
B_3	G_2	φ_2	$B_2 \oplus T$	$\varphi_1 + 2N$	$A_1 \oplus T$	
B_3	G_2	φ_2	D_3	$\varphi_1 + N$	A_2	
D_{n+1}	B_n	$\varphi_1 + N$	A_n	$\varphi_1 + \varphi_n$	A_{n-1}	$n > 2$
D_{n+1}	B_n	$\varphi_1 + N$	$A_n \oplus T$	$\varphi_1 + \varphi_n$	$A_{n-1} \oplus T$	$n > 2$
D_{2n}	B_{2n-1}	$\varphi_1 + N$	C_n	$\varphi_1 + \varphi_1$	C_{n-1}	$n > 1$
D_{2n}	B_{2n-1}	$\varphi_1 + N$	$C_n \oplus T$	$\varphi_1 + \varphi_1$	$C_{n-1} \oplus T$	$n > 1$
D_{2n}	B_{2n-1}	$\varphi_1 + N$	$C_n \oplus A_1$	$\varphi_1 + \varphi_1$	$C_{n-1} \oplus A_1$	$n > 1$
D_8	B_7	$\varphi_1 + N$	B_4	φ_4	B_3	
D_4	B_3	φ_3	B_2	$\varphi_1 + 3N$	A_1	
D_4	B_3	φ_3	$B_2 \oplus T$	$\varphi_1 + 3N$	$A_1 \oplus T$	
D_4	B_3	φ_3	$B_2 \oplus A_1$	$\varphi_1 + 3N$	$A_1 \oplus A_1$	
D_4	B_3	φ_3	D_3	$\varphi_1 + 2N$	A_2	
D_4	B_3	φ_3	$D_3 \oplus T$	$\varphi_1 + 2N$	$A_2 \oplus T$	
D_4	B_3	φ_3	B_3	$\varphi_1 + N$	G_2	

TABLE 1

Remark 1. In Table 1 above N stands for the trivial representation. The types of simple Lie algebras are denoted as usual, following [17].

3 Proof of Theorem 3

3.1 Preliminary statements

Proposition 2. *Let $P(M, G)$ be a principal fiber bundle with a connection form θ and the curvature form Ω . Assume that $M = K/H$ is a homogeneous space of a compact Lie group K . Fix a point $p \in P$. Let \mathfrak{h}' be the Lie subalgebra of \mathfrak{g} generated by all $\Omega_p(X, Y)$, $X, Y \in \mathcal{H}_p$. Then, if there exists a fat associated bundle with fiber G/L , then $\mathfrak{g} = \mathfrak{h}' + \mathfrak{l}$. Moreover, if the bundle is fat with respect to the canonical connection, then $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$.*

Proof. Notice that the fatness condition is equivalent to the condition that all 2-forms $B_{\mathfrak{g}}(\Omega_p(X, Y), u)$ are non-degenerate for all $u \in \mathfrak{l}^\perp$, all $p \in P$ and $X, Y \in \mathcal{H}_p$. The latter implies $(\mathfrak{h}')^\perp \cap \mathfrak{l}^\perp = \{0\}$. But

$(\mathfrak{h}')^\perp \cap \mathfrak{l}^\perp = \{0\}$ iff $\mathfrak{g} = \mathfrak{h}' + \mathfrak{l}$. We complete the proof by noticing that $\mathfrak{h}' \subset \mathfrak{h}^*$, where \mathfrak{h}^* denotes the holonomy Lie algebra, and that for the canonical connection $\mathfrak{h}^* \subset \mathfrak{h} \cong \lambda(\mathfrak{h})$, by Theorem 5. \square

Let there be given a principal G -bundle $G \rightarrow P \rightarrow K/H$ endowed with a principal K -invariant connection θ . Use Theorem 4. On the Lie algebra level we get that a connection θ is given by a linear mapping $\Lambda : \mathfrak{k} \rightarrow \mathfrak{g}$ such that

- $\Lambda(X) = X$ for every $X \in \mathfrak{h}$,
- $\Lambda[X, Y] = [X, \Lambda(Y)]$ for every $X \in \mathfrak{h}$ and $Y \in \mathfrak{k}$.

Here we identify X with $\lambda(X)$. Let $L \subset G$ be a closed subgroup and $A := \mathfrak{l}^\perp \subset \mathfrak{g}$ be the orthogonal complement of \mathfrak{l} in \mathfrak{g} with respect to $B_{\mathfrak{g}}$ (where $B_{\mathfrak{g}}$ denotes an invariant, negative-definite 2-form on \mathfrak{g}). Let also $B := \mathfrak{h} \cap \mathfrak{l}$ and $C := (\mathfrak{h} \cap \mathfrak{l})_{\mathfrak{h}}^\perp \subset \mathfrak{h}$ be the orthogonal complement of $\mathfrak{h} \cap \mathfrak{l}$ in \mathfrak{h} with respect to $B_{\mathfrak{g}}$. Throughout this note we use the following notation: if $W = U \oplus V$, is a direct sum of vector spaces, the projection onto V is denoted by $proj_V$.

Lemma 1. *If $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ then for any subspace $V \subset \mathfrak{k}$ the conditions (1) and (2) below are equivalent*

$$\forall_{0 \neq X \in A} \forall_{0 \neq Y \in V} \exists_{0 \neq Z \in V} B_{\mathfrak{g}}(X, [Y, Z]_{\mathfrak{h}}) \neq 0. \quad (1)$$

$$\forall_{0 \neq X \in C} \forall_{0 \neq Y \in V} \exists_{0 \neq Z \in V} B_{\mathfrak{g}}(X, [Y, Z]_{\mathfrak{h}}) \neq 0. \quad (2)$$

Proof. We have the (orthogonal) direct sum decompositions $\mathfrak{h} = B \oplus C$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ and thus $\mathfrak{g} = B \oplus C \oplus \mathfrak{h}^\perp$. For any vector $X \in \mathfrak{g}$, denote its A -component (B , C -component, etc) by X_A (X_B , X_C , etc). Note that for $X \in A \subset \mathfrak{g}$ and $H \in \mathfrak{h}$ we have $H = H_B + H_C$ and

$$\begin{aligned} B_{\mathfrak{g}}(X, H) &= B_{\mathfrak{g}}(X, H_B + H_C) = B_{\mathfrak{g}}(X, H_C) = B_{\mathfrak{g}}(X_{\mathfrak{h}^\perp} + X_{\mathfrak{h}}, H_C) = \\ B_{\mathfrak{g}}(X_{\mathfrak{h}}, H_C) &= B_{\mathfrak{g}}(X_{\mathfrak{h}}^B + X_{\mathfrak{h}}^C, H_C) = B_{\mathfrak{g}}(X_{\mathfrak{h}}^C, H_C) = B_{\mathfrak{g}}(X_C, H_C) = \\ &= B_{\mathfrak{g}}(X_C, H_C + H_B) = B_{\mathfrak{g}}(X_C, H) \end{aligned}$$

where $X_{\mathfrak{h}}^B$ ($X_{\mathfrak{h}}^C$) is the B -component (C -component) of $X_{\mathfrak{h}}$.

Thus the condition (1) implies that

$$\text{Ker}(proj_C)|_A = \{0\} \quad (3)$$

But

$$\mathfrak{g} = \mathfrak{l} \oplus A = \mathfrak{l} \oplus C$$

are the direct sum decompositions and therefore $\dim A = \dim C$, so $proj_C|_A : A \rightarrow C$ is an isomorphism. Thus,

$$B_{\mathfrak{g}}(X, [Y, Z]_{\mathfrak{h}}) = B_{\mathfrak{g}}(X_C, [Y, Z]_{\mathfrak{h}}) = B_{\mathfrak{g}}(proj_C(X), [Y, Z]_{\mathfrak{h}})$$

therefore condition (1) implies condition (2)..

Now we will show that condition (2) implies condition (1). Assume that (1) is not satisfied. Then there exists $0 \neq X \in A$ and $0 \neq Y \in V$ such that

$$\forall 0 \neq Z \in V \quad B_{\mathfrak{g}}(X, [Y, Z]_{\mathfrak{h}}) = 0$$

therefore

$$\forall 0 \neq Z \in V \quad B_{\mathfrak{g}}(X_C, [Y, Z]_{\mathfrak{h}}) = 0.$$

If $X_C \neq 0$ then the lemma is proved. Assume that $X_C = 0$. As $X \in A$ thus

$$X = X_C + X_{\mathfrak{h}^\perp} = X_{\mathfrak{h}^\perp},$$

that is $0 \neq X \in X_{\mathfrak{h}^\perp}$, so X is a non-zero vector which is orthogonal to \mathfrak{h} and to \mathfrak{l} . This is impossible since $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$. \square

Lemma 2. *Let $\Lambda_{\mathfrak{h}} : \mathfrak{k} \rightarrow \mathfrak{h}$ be the linear operator defined by*

$$\Lambda_{\mathfrak{h}}(X) := proj_{\mathfrak{h}} \circ \Lambda(X).$$

The map $\Lambda_{\mathfrak{h}}$ induces an invariant connection in the principal bundle $K \rightarrow K/H$. The associated bundle

$$G/L \rightarrow P/L \rightarrow K/H$$

is fat (with respect to an invariant connection) if and only if the bundle

$$H/(H \cap L) \rightarrow K/L \rightarrow K/H$$

is fat.

Proof. Note that in the proof of this Lemma, for brevity, we use a terminology which is somewhat not precise but clear: we say that Λ or $\Lambda_{\mathfrak{h}}$ is fat, if it yields a fat associated bundle in our usual sense. The condition $\Lambda_{\mathfrak{h}}(X) = X$ for $X \in \mathfrak{h}$ is obviously satisfied. Also

$$\Lambda([X, Y]) = \Lambda_{\mathfrak{h}}([X, Y]) + \Lambda_{\mathfrak{h}^\perp}([X, Y])$$

and

$$[X, \Lambda(Y)] = [X, \Lambda_{\mathfrak{h}}(Y) + \Lambda_{\mathfrak{h}^\perp}(Y)] = [X, \Lambda_{\mathfrak{h}}(Y)] + [X, \Lambda_{\mathfrak{h}^\perp}(Y)],$$

for $X \in \mathfrak{h}$ and $Y \in \mathfrak{k}$. But $[X, \Lambda_{\mathfrak{h}}(Y)] \in \mathfrak{h}$ and $[X, \Lambda_{\mathfrak{h}^\perp}(Y)] \in \mathfrak{h}^\perp$ since K/H is reductive and, therefore, we obtain

$$\Lambda_{\mathfrak{h}}([X, Y]) = [X, \Lambda_{\mathfrak{h}}(Y)],$$

as desired.

It follows from Lemma 1 that if Λ is fat then

$$\forall_{0 \neq X \in (\mathfrak{h} \cap \mathfrak{l})_{\mathfrak{h}}^\perp} \quad \forall_{0 \neq Y \in \text{Ker } \Lambda} \quad \exists_{Z \in \text{Ker } \Lambda} \quad B_{\mathfrak{g}}(X, \Lambda([Y, Z])) \neq 0.$$

But

$$\begin{aligned} B_{\mathfrak{g}}(X, \Lambda([Y, Z])) &= B_{\mathfrak{g}}(X, \Lambda_{\mathfrak{h}}([Y, Z]) + \Lambda_{\mathfrak{h}^\perp}([Y, Z])) = \\ &= B_{\mathfrak{g}}(X, \Lambda_{\mathfrak{h}}([Y, Z])) + B_{\mathfrak{g}}(X, \Lambda_{\mathfrak{h}^\perp}([Y, Z])) = B_{\mathfrak{g}}(X, \Lambda_{\mathfrak{h}}([Y, Z])) \end{aligned}$$

and so if Λ is fat then

$$\forall_{0 \neq X \in (\mathfrak{h} \cap \mathfrak{l})_{\mathfrak{h}}^\perp} \quad \forall_{0 \neq Y \in \text{Ker } \Lambda} \quad \exists_{Z \in \text{Ker } \Lambda} \quad B_{\mathfrak{g}}(X, \Lambda_{\mathfrak{h}}([Y, Z])) \neq 0.$$

On the other hand if $\Lambda_{\mathfrak{h}}$ is fat then again by Lemma 1 it is fat as a map $\mathfrak{k} \rightarrow \mathfrak{g}$. \square

Now we are ready to get the main part of the proof of Theorem 3.

Lemma 3. *Let $G \rightarrow P \rightarrow K/H$ be a G -structure over a homogeneous space K/H of a compact Lie group K . Assume that G is compact. An associated bundle*

$$G/L \rightarrow P/L \rightarrow K/H$$

is fat with respect to a connection metric determined by the K -invariant canonical connection if and only if the following three conditions are satisfied:

1. $G = HL$,
2. K/H is a Riemannian symmetric space;
3. the pair $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{l})$ is a Bérard-Bergery pair.

Proof. By proposition 2, we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$. It follows from Lemma 2 that

$$G/L \rightarrow P/L \rightarrow K/H$$

is fat if and only if the homogeneous bundle $H/H \cap L \rightarrow K/L \rightarrow K/H$ is fat, hence, K/H must be symmetric by Proposition 1, the triple (K, H, L) must be the Bérard-Bergery triple, and the pair $(H, H \cap L)$ must be the Bérard-Bergery pair. \square

3.2 Proof of Theorem 3

We complete the proof in two steps. Assume, first, that G is simple. If G is simple then $G/L \rightarrow P/L \rightarrow K/H$ is fat if and only if it is Bérard-Bergery bundle. Indeed, by Lemma 3 $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ and the pair $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{l})$ is a Bérard-Bergery pair. Thus, either $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ is a factorization or $\mathfrak{g} = \mathfrak{h}$. In the first case, the triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ and $\mathfrak{h} \cap \mathfrak{l}$ are contained in Table 1. However, the comparison of Table 1 and Table 2 shows that none of the pairs $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{l})$ is contained in both tables. Hence, $G = H$. We get the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{=} & H \\ \downarrow & & \downarrow \\ K & \xrightarrow{i} & P \\ \downarrow & & \downarrow \\ K/H & \xrightarrow{=} & K/H \end{array}$$

where i denotes the evaluation map. Clearly, under the conditions of Theorem 3, we get an isomorphism of the principal bundles. This follows since the isotropy representation is faithful, and, therefore, i must be an embedding, hence a diffeomorphism. This completes the proof when G is simple. Note that the argument also goes through for any $\mathfrak{g} = \mathfrak{h}$. In the general case, the proof follows from Lemma 4 below. We need to consider the case when \mathfrak{g} is not simple, but $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ is a factorization.

Lemma 4. *Under the assumptions of Theorem 3, the following holds*

$$\mathfrak{g} = \mathfrak{h}' + \mathfrak{l}, \quad \mathfrak{g} = \mathfrak{h}' + \mathfrak{n} \text{ and } \mathfrak{l} = \mathfrak{h}' + \mathfrak{n}.$$

Proof. We know that $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ and $\mathfrak{h} = \mathfrak{h}' + \mathfrak{n}'$. But $\mathfrak{n}' \subset \mathfrak{h} \cap \mathfrak{l}$ and therefore

$$\mathfrak{g} = \mathfrak{h}' + \mathfrak{l}.$$

It follows from Table 2 that \mathfrak{h}' is a simple Lie algebra or $\mathfrak{h}' = \mathbb{R} + \mathfrak{su}(2)$. Take a connected Lie subgroup H' corresponding to \mathfrak{h}' (it exists if \mathfrak{h}' is simple; if \mathfrak{h}' is not simple then \mathbb{R} corresponds to a circle and $H'/(H \cap L)$ is not simply connected). So assume that H' is simple. In this case it follows from Table 2 and (see comments to Corollary to Theorem 5.1 and Corollary to Theorem 6.1 in [15]) that only H' acts effectively on $H'/(H \cap L)'$ and therefore $\mathfrak{g} = \mathfrak{h}' + \mathfrak{n}$ and $\mathfrak{l} = \mathfrak{h}' + \mathfrak{n}$ for some ideal $\mathfrak{n} \subset \mathfrak{g}$. \square

Now, we complete the proof of Theorem 3 by applying Lemma 3 and Lemma 4.

4 Addendum: the Bérard-Bergery pairs

4.1 Table 2

For the convenience of the reader we reproduce a part of the Bérard-Bergery classification [3] (see explanations below Table 2).

\mathfrak{h}	$\mathfrak{h} \cap \mathfrak{l}$	restrictions
$A_{n-2} \oplus \mathbb{R} \oplus A_1$	$A_{n-2} \oplus \mathbb{R} \oplus \mathbb{R}_1$	$n \geq 2$
$B_{n-2} \oplus A_1 \oplus A_1$	$B_{n-2} \oplus A_1 \oplus \mathbb{R}_1$	$n \geq 2$
$C_{n-1} \oplus A_1$	$C_{n-1} \oplus \mathbb{R}_1$	$n \geq 2$
$D_{n-2} \oplus A_1 \oplus A_1$	$D_{n-2} \oplus A_1 \oplus \mathbb{R}_1$	$n \geq 3$
$A_5 \oplus A_1$	$A_5 \oplus \mathbb{R}_1$	
$D_6 \oplus A_1$	$D_6 \oplus \mathbb{R}_1$	
$E_7 \oplus A_1$	$E_7 \oplus \mathbb{R}_1$	
$C_3 \oplus A_1$	$C_3 \oplus \mathbb{R}_1$	
$\overline{A}_1 \oplus A_1$	$\overline{A}_1 \oplus \mathbb{R}_1$	
$A_1 \oplus \overline{A}_1$	$A_1 \oplus \mathbb{R}_1$	
$A_{n-4} \oplus \mathbb{R} \oplus A_3$	$A_{n-4} \oplus \mathbb{R} \oplus \overline{C}_2$	$n \geq 4$
$B_{n-4} \oplus D_4$	$B_{n-4} \oplus \overline{B}_3$	$n \geq 4$
$C_{n-2} \oplus C_2$	$C_{n-2} \oplus A_1 \oplus A_1$	$n \geq 3$
$D_{n-4} \oplus D_4$	$D_{n-4} \oplus \overline{B}_3$	$n \geq 5$
$\mathbb{R} \oplus D_5$	$\mathbb{R} \oplus B_4$	
$A_1 \oplus D_6$	$A_1 \oplus B_5$	
D_8	B_7	
B_4	D_4	
$A_{n-2} \oplus \mathbb{R} \oplus A_1$	$A_{n-2} \oplus \mathbb{R}_a$	$n \geq 2$
$\mathbb{R} \oplus A_1 \oplus A_1$	$A_1 \oplus \mathbb{R}_a$	

TABLE 2

4.2 Explanations

We write down only the Bérard-Bergery pairs $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{l})$ which come from the *semisimple* Lie algebras \mathfrak{k} . The symbols A_1 and \bar{A}_1 denote subalgebras associated to the highest and the lowest root. Note that our notation differs from that of [3]: the triples $(\mathfrak{k}, \mathfrak{h}, \mathfrak{h} \cap \mathfrak{l})$ are denoted by $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k})$ in [3]. Also, we have reproduced only cases (A), (C) and (D) from the classification. Cases (B), (E) and (F) are not essential in our considerations. This follows from the description of the classification in [3]:

1. case (B) is obtained from case (A) by removing \mathbb{R}_1 from $\mathfrak{h} \cap \mathfrak{l}$
2. case (E) consists of the pairs

$$(A'_1 \oplus \mathfrak{h}' \oplus A''_1 \oplus \mathfrak{h}'', \bar{A}_1 \oplus \mathfrak{h}' \oplus \mathfrak{h}''),$$

where $(A'_1 \oplus \mathfrak{h}', \mathfrak{h}')$ is of type (B), and $(A''_1 \oplus \mathfrak{h}'', \mathfrak{h}'')$ is either of type (B), or $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{l})$ has the form $(A_1 \oplus A_1, \bar{A}_1)$, where \bar{A}_1 is a diagonal subalgebra in $A_1 \oplus A_1$.

3. case (F) is eliminated by the assumption that \mathfrak{k} is semisimple,
4. the dual triples in case (C) yield the same pair $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{l})$ (see Lemma 12 in [3]).

Remark 2. We consider only the case of the canonical connection in a G -structure. We conjecture that the same holds for any invariant connection, but the problem seems to be much more difficult. It would be interesting to try some other better known invariant connections, for example, metric connections described in spirit of [1].

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